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The meaning of infrared singularities in noncommutative gauge theories

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Abstract: We point out that the leading infrared singular terms in the effective actions of noncommutative gauge theories arising from nonplanar loop diagrams have a natural interpretation in terms of the matrix model (operator) formulation of these theories. In this formulation (for maximal spatial noncommutativity), noncommutative space arises as a configuration of an infinite number of D-particles. We show that the IR singular terms correspond to instantaneous linear potentials between these D-particles resulting from the zero point energies of fluctuations about this background. For theories with fewer fermionic than bosonic degrees of freedom, such as pure noncommutative gauge theory, the potential is attractive and renders the theory unstable. With more fermionic than bosonic degrees of freedom, the potential is repulsive and we argue that the theory is stable, though oddly behaved.

Keywords: D-branes, M(atrix) Theories, Non-Commutative Geometry.

JHEP11(2001)006

Contents

JHEP11(2001)006

1. Introduction

In the context of string theory, noncommutative field theories [1, 2, 3] appear as a description of the low energy physics of D-branes in the presence of constant B (NS-NS two-form) fields, or equivalently, D-branes carrying uniform lower-dimensional brane charges. Alternatively, we may describe the situation directly in terms of these lower dimensional branes carrying the higher dimensional brane charge. For example, a noncommutative D2-brane with a uniform zero-brane charge density may be described as a collection of an infinite number of D0-branes whose (infinite dimensional) configuration matrices realize the algebra of the noncommutative plane, $[X^1, X^2] = i\theta$. The matrices describing the fluctuations about this background may be expanded in a basis generated by the background matrices (noncommutative coordinates) X^1 and X^2; in other words they are fields on the noncommutative plane. Thus, noncommutative field theory is the natural description arising in terms of the lower dimensional brane degrees of freedom.[1]

[1] The Seiberg-Witten map [3] between noncommutative and commutative descriptions may be understood as the map between the lower dimensional brane degrees of freedom and those of the higher dimensional brane.

While noncommutative field theories outside the context of string theory are not obviously related to D-branes, there typically exists a matrix model description (the operator formalism) analogous to the lower dimensional brane picture in string theory. In particular, for pure noncommutative gauge theory, the relevant matrix model is just a bosonic version of the D0-brane matrix model with the number of scalars equal to the number of noncommutative dimensions. Thus, even outside the context of string theory one may think of noncommutative gauge theory as describing fluctuations about a particular state of an infinite number of (bosonic) D-particles. In this paper, we show that certain intriguing features of quantum noncommutative gauge theories can be understood naturally in terms of this D-particle picture.

Perhaps the most surprising feature of quantum noncommutative field theories is the phenomenon of UV-IR mixing. As demonstrated in [4] for noncommutative scalar theories, high momentum virtual particles running in loops of nonplanar diagrams can lead to long range correlations. In particular, even for a massive theory, nonplanar diagrams give contributions to the effective action which exhibit infrared singularities. As demonstrated in [5, 6], these infrared singular terms appear also in the effective action of noncommutative gauge theories. For example, in four dimensions, nonplanar one loop diagrams contribute a term proportional to

$$g^2 A_\mu(p) A_\nu(-p) \frac{\tilde{p}^\mu \tilde{p}^\nu}{|\tilde{p}|^4}\,, \tag{1.1}$$

where $\tilde{p}^\mu = \theta^{\mu\nu} p_\nu$. This term is singular as $p \to 0$ and will clearly have dramatic consequences for the long distance physics of the theory.

The main result of this note is to demonstrate that the term (1.1) corresponds to physics that is actually quite familiar when viewed in terms of the D-particle picture. Indeed, it is simply an instantaneous linear potential between the D-particles making up the noncommutative space. If not for supersymmetry, such a potential would exist between D0-branes in string theory due to the zero-point energies of the strings connecting them. The famous v^4/r^7 potential between widely separated D0-branes in the BFSS matrix model [7] arises only after cancellation between contributions from off-diagonal bosonic degrees of freedom and contributions from off diagonal fermionic degrees of freedom, both of which have linear potentials at leading order [8]. Thus, an instantaneous linear potential between nonsupersymmetric D-particles is completely expected, with an attractive sign for theories with more bosonic than fermionic degrees of freedom and with a repulsive sign for theories with an excess of fermionic degrees of freedom.

In the attractive case, we show that the one loop potential is unbounded below (relative to the energy of the chosen vacuum) and renders the theory unstable. Based on the D-particle intuition, we expect this instability to persist nonperturbatively. Thus it appears that pure noncommutative gauge theory is not a sensible quantum field theory. On the other hand, with an excess of fermionic degrees of freedom,

JHEP11(2001)006

the D-particle repulsion leads to a positive definite potential and a "confinement" of density fluctuations, so the theory appears to be stable (with appropriate boundary conditions at infinity).

The plan for the remainder of the paper is as follows. In section 2, we write down the leading infrared singular quadratic term in the one-loop effective action for d-dimensional noncommutative gauge theory. This term is gauge invariant at leading order in the noncommutativity parameter but not under the full noncommutative gauge transformation. In section 3, we suggest a minimal gauge invariant completion of this leading quadratic term, involving a two point function of the simplest gauge invariant open Wilson line operator. In section 4, we briefly review the matrix model formulation of the theory and then recast our proposed gauge invariant effective action in terms of this D-particle picture in section 5, showing that it is exactly an instantaneous linear potential between the D-particles. A direct derivation of this potential from the matrix model is included in an appendix. In section 6, we discuss the physical consequences of our observation, arguing that pure noncommutative gauge theory (and theories with fewer bosonic than fermionic degrees of freedom) is unstable due to the attractive linear potential, while theories with an excess of fermionic degrees of freedom are stable. A few concluding remarks are offered in section 7.

The literature on noncommutative field theories is by now very large. For recent reviews of the subject, including large lists of references, we refer the reader to [9, 10, 11]. For an interesting previous discussion of infrared singularities and other properties of noncommutative field theories in terms of the matrix model formalism, see [12].

2. Nonplanar effective action in noncommutative gauge theory

In this section, we write down the infrared singular quadratic terms in the effective action of d-dimensional noncommutative gauge theory arising from non-planar diagrams. These terms were originally calculated for 4 dimensional gauge theory in [5, 6].

We consider the action for a $U(N)$ gauge field in d-dimensional Minkowski space-time, with fermions ψ_j and scalars ϕ_i, all in the adjoint representation,[2]

$$S = \int d^dx \; \mathrm{tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \phi_i D^\mu \phi_i - \frac{1}{2} m_i^2 \phi_i^2 - i\bar{\psi}_j (\gamma^\mu D_\mu + m_j) \psi_j \right) ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]_\star$ and $D_\mu = \partial_\mu + ig[A_\mu, \;]_\star$. The action is invariant under a noncommutative gauge transformation

$$\delta A_\mu = \partial_\mu \Lambda + ig[A_\mu, \Lambda]_\star \qquad \delta\phi = -ig[\phi, \Lambda]_\star \qquad \delta\psi = -ig[\psi, \Lambda]_\star \,. \tag{2.1}$$

[2]We ignore the possibility of fundamental matter fields since these only give rise to planar diagrams.

It is straightforward to compute the quadratic terms in the one-loop, 1PI effective action for the gauge field, and it was found [5, 6] that nonplanar diagrams give rise to new infrared singular terms of the form

$$\int d^4p \,\mathrm{tr}\,(A_\mu(p))\mathrm{tr}\,(A_\nu(-p))\frac{\tilde{p}^\mu\tilde{p}^\nu}{|\tilde{p}|^4}$$

in the four dimensional case, where $\tilde{p}^\mu = \theta^{\mu\nu}p_\nu$. In appendix A, we find that in d-dimensions the nonplanar quadratic effective action is

$$\Gamma_{1PI} = \int \frac{d^dp}{(2\pi)^d}\mathrm{tr}\,(A_\mu(p))\mathrm{tr}\,(A_\nu(-p))\frac{\tilde{p}^\mu\tilde{p}^\nu}{|\tilde{p}|^d}\left\{\frac{g^2}{2\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)\left(d-2+N_s-\frac{1}{2}\mathcal{N}_f\right)\right\}, \tag{2.2}$$

where N_s and $\mathcal{N}_f$ are the number of scalars and the number of fermionic degrees of freedom respectively. As pointed out in [4, 6], the leading infrared singularities cancel if there are an equal number of bosonic and fermionic degrees of freedom. With an excess of bosonic degrees of freedom, we find that the nonplanar effective potential (V_{NP} where $\Gamma_{NP} = \int -V_{NP}$) is negative, suggesting tachyonic behavior for the low momentum modes of the gauge field. For an excess of fermionic degrees of freedom, the effective potential is positive, so the infrared singular term does not appear to cause any instability. This behavior is opposite to that which was suggested previously in [6], due to a discrepancy in the sign of the term (2.2).[3] Shortly, we will develop an intuitive understanding of the origin of this term and present an independent argument that the present sign is correct. We leave further discussion of the consequences of this term until then.

3. Gauge invariant effective action

It is easy to check that the quadratic term we have calculated is invariant under a noncommutative gauge transformation (2.1) to leading order in θ, since upon the variation $\delta A_\mu = ip_\mu\Lambda$, our effective action cancels due to the vanishing of $p_\mu\tilde{p}^\mu = \theta^{\mu\nu}p_\mu p_\nu$. On the other hand, it is not invariant under the full noncommutative gauge transformation. Since the nonlinear term in the gauge transformation mixes terms with different numbers of occurrences of the gauge field, it is reasonable to expect that terms in the effective action at higher orders in the gauge field will complete (2.2) into a gauge invariant expression. This was argued to be the case for the effective action of the 3+1 dimensional noncommutative $\mathcal{N} = 4$ theory on its Coulomb branch in [14, 15].

Since matrix products always occur together with star products in the original action, the double trace structure of (2.2) suggests that the one-loop nonplanar gauge

[3] A discrepancy with [6] has been found previously by other authors, including [23].

invariant effective action should take the form

$$\int d^dp \sum_i W_i(p)\tilde{W}_i(-p)\Delta_i(p)\,, \tag{3.1}$$

where $\Delta_i(p)$ are some functions of momenta and W_i and $\tilde{W}_i$ are operators that are separately gauge invariant, possibly with some indices contracted with indices on Δ_i. Gauge invariant operators with non-zero momenta in noncommutative gauge theory generically include open Wilson lines [13], and take the form

$$W(p) = \int d^dx e^{ip\cdot x}\mathrm{tr}\,\left(\mathcal{O}\star \mathrm{P}_\star e^{ig\int_x^{x+\tilde{p}} A}\right)\,, \tag{3.2}$$

where the path-ordered Wilson line runs over the straight line path from x to $x+\tilde{p}$ and $\mathcal{O}$ is some gauge covariant operator built from field strengths and covariant derivatives.[4] The simplest such operator has $\mathcal{O}=1$, and has an expansion in powers of A given by

$$\begin{aligned} \omega(p) &= \int d^dx e^{ip\cdot x}\mathrm{tr}\,\left(\mathrm{P}_\star e^{ig\int_x^{x+\tilde{p}} A}\right) \\ &= (2\pi)^d N\delta(p) + ig\ \mathrm{tr}\ (A_\mu(p))\,\tilde{p}^\mu + \mathcal{O}(A^2)\,. \end{aligned} \tag{3.3}$$

Note that the term linear in A has exactly the structure appearing in the quadratic effective action. In particular, a term

$$\int d^dp\ \omega(p)\omega(-p)\frac{1}{|\tilde{p}|^d} \tag{3.4}$$

is gauge invariant and has a leading term with precisely the form (2.2).[5] The only other operator of the form (3.2) that yields a term linear in A with the same structure is the one with $\mathcal{O}=\theta^{\mu\nu}F_{\mu\nu}$. Calling this operator $A(p)$, one could also write terms

$$\int d^dpA(p)\omega(-p)\frac{1}{|\tilde{p}|^d}\,, \qquad \int d^dpA(p)A(-p)\frac{1}{|\tilde{p}|^d}$$

that reproduce the structure (2.2). However, both of these produce cubic terms in the effective action (coming from the nonlinear term in F) that are more singular than the cubic terms obtained by a direct calculation. Thus, it seems that the quadratic term (2.2) in the effective action must arise from the gauge invariant structure (3.4). In the next sections, we will see that this term has a very natural interpretation in terms of the matrix model formulation of the gauge theory. Note that we expect the full one-loop effective action to contain additional terms (probably an infinite series) of the form (3.1), but all of them will be less important than (3.4) for small momenta.

[4] More generally, we may insert a series of covariant operators at various points along the Wilson line and consider a Wilson line of arbitrary shape whose net displacement is $\tilde{p}$.

[5] The leading delta function term in ω may be ignored since it only contributes to an infinite constant term in the potential.

4. Review of the matrix model formulation

We now recall the matrix model formulation of $\mathrm{U}(N)$ noncommutative gauge theory, focusing on the case of pure gauge theory with maximal rank spatial noncommutativity (θ of rank $2p$ in $2p$ spatial dimensions). We begin with the action for a system of bosonic D0-branes in $2p$ dimensions,

$$S = \int dt \,\mathrm{Tr}\left(\frac{1}{2}\dot{X}^i\dot{X}^i + \frac{1}{4}[X^i, X^j]^2\right).$$

For an infinite number of branes, there exists a classical solution $X^i = x^i$ to the equations of motion such that

$$[x^i, x^j] = i\theta^{ij} \otimes \mathbb{1}_{N\times N}\,.$$

For example, we may choose $\sqrt{\frac{\theta}{2}}(x^{2n-1} + ix^{2n}) = a_n \otimes \mathbb{1}_{N\times N}$ where a_n are matrix representations of a set of p harmonic oscillator creation operators.[6] We may now expand the matrix theory action about this background, defining $X^i = x^i + \theta^{ij}A_j$. We find (see e.g. [16])

$$S = \int \frac{d^{2p+1}x}{(2\pi)^p \mathrm{Pf}(\theta)} \mathrm{tr}\left(\frac{1}{2}\dot{A}^i\dot{A}^i - \frac{1}{4}G^{km}G^{ln}\{F_{kl} - \theta^{-1}_{kl}\}\{F_{mn} - \theta^{-1}_{mn}\}\right), \tag{4.1}$$

where $G^{ij} \equiv \theta^{ik}\theta^{kj}$ and F is the noncommutative field strength. Here, we have made the transition from the matrix theory formalism to the field theory formalism by considering A to be a function generated by the noncommutative coordinates x^i and making the usual substitutions

$$[x_i,\ ] \to i\theta^{ij}\partial_j\,, \qquad \mathrm{tr}\,(f_1\cdots f_n) \to \int \frac{d^{2p}x}{(2\pi)^p\mathrm{Pf}(\theta)}\mathrm{tr}\,(f_1 \star \cdots \star f_n)\,.$$

The action (4.1) is precisely the action for pure noncommutative gauge theory in $2p+1$ dimensions in the gauge $A_0 = 0$ (after an appropriate redefinition of the coordinates such that $G^{ij} \to \delta^{ij}$). We may ignore the θ^{-1} terms since $\int \mathrm{tr}\,(\theta^{-1}F)$ vanishes for allowed fluctuations about the perturbative vacuum.

5. Interpretation of the nonplanar effective action

We would now like to interpret the term (3.4) in the nonplanar effective action in the context of the matrix model. As reviewed in appendix B, the open Wilson line operator $\omega(k)$ appearing in (3.4) takes a very simple form in the matrix model language, namely

$$\omega(k) = (2\pi\theta)^p\,\mathrm{Tr}\,(e^{ik\cdot X}) \equiv (2\pi\theta)^p\rho(k)\,.$$

[6] We choose θ to take the standard form $\theta^{ij} = \theta(i\sigma_2)\otimes \mathbb{1}_{p\times p}$.

This operator also has an equally simple interpretation. In type IIA sting theory, $\rho(k)$ is precisely the leading operator coupling to $C_0(k)$, the time component of the RR one-form [19]. Therefore, $\rho(k)$ is the operator that measures the density of zero-brane charge. Given this interpretation, it is natural to rewrite the effective action (2.2) in position space, and we find

$$\Gamma_{NP} = -\frac{1}{2}\left(d - 2 + N_s - \frac{1}{2}\mathcal{N}_f\right)\int dt \int d^{2p}x \int d^{2p}y \; \rho(x,t)\rho(y,t)|x-y|\,, \qquad (5.1)$$

where the position space density operator is

$$\rho(x,t) = \int \frac{d^{2p}k}{(2\pi)^{2p}}\, e^{-ik\cdot x}\; \mathrm{Tr}\left(e^{ik\cdot X(t)}\right).$$

Thus, the infrared singular terms in the nonplanar effective action correspond to an instantaneous linear potential between the D-particles making up the noncommutative space.

This potential is exactly what one should have expected for bosonic D-particles. Recall that the success of the BFSS matrix model in describing nine large spatial dimensions of DLCQ M-theory depended crucially on the presence of supersymmetries to ensure that the classical flat directions were not lifted by quantum mechanical zero point energies. For the purely bosonic theory, there is no cancellation, and any widely separated pair of D-particles experience an attractive linear potential from the string connecting them. With an excess of fermionic degrees of freedom, the attractive potential will be over-canceled and we will have a repulsive linear potential between the zero-branes.

As a check of this interpretation, we perform a direct calculation in appendix C of the one loop effective action in the matrix theory formulation by explicitly summing the zero point energies of fluctuating modes. We find that the leading term in the matrix theory effective action reproduces the complete expression (5.1) exactly, including the coefficient.

The matrix theory calculation and the intuitive picture we have developed also provide checks of the sign of the potential (2.2) calculated in the field theory formalism. It is straightforward to verify that the negative effective potential in (2.2) found for bosonic gauge theories corresponds to an attractive D-particle potential, while the positive potential occurring in theories with excess fermions corresponds to a repulsive potential. In the next section we will explain why the attractive potential leads to an instability while the repulsive potential does not.

6. Physical consequences

In this section, we consider the physical consequences of the potential (5.1). At first sight, instantaneous linear forces between the D-particles would seem to be disastrous for the theory. One might expect that the space would simply collapse in the presence

JHEP11(2001)006

of attractive forces and blow up in the presence of repulsive forces. However, one must be careful to specify what types of fluctuations are allowed, and in particular, what boundary conditions to impose at infinity.

In field theory, it is usual to require that physical fields fall off sufficiently rapidly at infinity. In our case, a natural condition would be to require that the zero-brane density remains fixed and uniform at infinity. In the matrix theory formalism, this should correspond to a restriction to fluctuation matrices A of finite rank or some norm completion of this set.[7] With such a restriction, uniform expansion or collapse of the space is not allowed since it alters the fields at infinity. We now turn to the allowed localized fluctuations.

With a fixed uniform density ρ_0 at infinity, conservation of zero-brane charge will require that $\Delta = \rho - \rho_0$ integrates to zero over the space for fluctuations about the perturbative vacuum,

$$\int d^{2p}x \Delta(x) = 0 \,.$$

Thus, up to an infinite constant term (the energy of the initial uniform configuration), we may replace ρ with Δ in (5.1). This corresponds to subtracting off the δ-function term in the Wilson line operator (3.3).[8] The possible values of the resulting potential

$$V = C \int d^{2p}x \int d^{2p}y \Delta(x)\Delta(y)|x-y|$$

are more clear after a Fourier transform to momentum space, which gives

$$\begin{aligned} V &= -C(2p-1)!! \int \frac{d^{2p}k}{(2\pi)^p} \Delta(k)\Delta(-k) \frac{1}{|k|^{2p+1}} \\ &= -C(2p-1)!! \int \frac{d^{2p}k}{(2\pi)^p} \frac{|\Delta(k)|^2}{|k|^{2p+1}} \,. \end{aligned}$$

In this form, it is clear that the potential is negative definite and unbounded below for the attractive sign ($C > 0$) and positive definite for repulsive D-particles ($C < 0$). We may understand this as follows.

Consider a small, local density fluctuation. By charge conservation, there must exist regions with positive Δ as well as regions with negative Δ. For the repulsive D-particle forces, these regions will effectively attract each other with a linear potential and the state of uniform density will tend to be restored. This “confinement” of density fluctuations results in the positive definite potential we have observed. On the other hand, with an attractive D-particle potential, regions of positive and negative density will tend to repel each other, and we can decrease the energy by an arbitrary amount by moving the positive and negative regions away from each other (see

[7]It would be valuable to understand the appropriate restriction more precisely.

[8]It is obvious that the remaining terms in the Wilson line operator vanish for $k = 0$, (i.e. $\Delta(k = 0) = 0$) so the claim that $\Delta(x)$ integrates to zero is justified.

JHEP11(2001)006

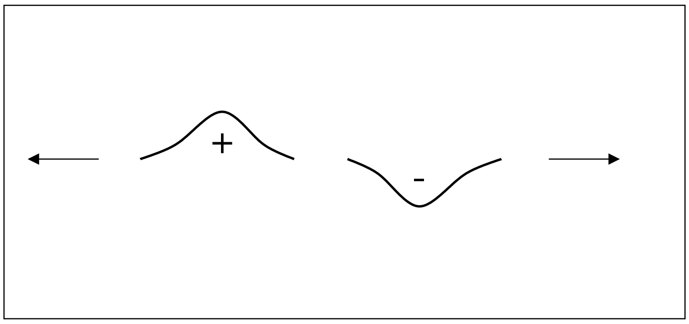

Figure 1: Repulsion of positive and negative density fluctuations leads to instability of noncommutative gauge theory.

figure 1), or by increasing the magnitude of the density fluctuations. Thus, while the space is prevented from collapsing by boundary conditions, the theory is still unstable due to local bunching up of the D-particles. Nonperturbatively, the preferred vacuum of the matrix model is really one where all of the D-particles sit together in a clump, so it seems unlikely that the instability we have observed will terminate at any stable extended configuration of D-particles.

The discussion in the previous paragraph ignored the tree level potential $-[X_i, X_j]^2$. This gives rise to the $\int F_{ij}F_{ij}$ potential in the field theory formalism and is clearly positive definite. Physically, it serves to suppress density fluctuations, however it is not enough to stabilize the bosonic theory. To see this, note that since it corresponds to a local term in the field theory action, its value for configurations with separated positive and negative density regions (such as the one depicted in figure 1) will be independent of the separation of the regions. Thus, the linear decrease in energy as the two regions are separated persists and the total potential is still unbounded below.

While the repulsive theories appear to be stable, they will certainly have some very unusual properties, both because the nonplanar potential is instantaneous and because it is so strong. It would be interesting to understand these theories better based on the D-particle intuition but we leave this as a problem for future work.

Supersymmetric theories

The D-particle intuition can also help us to understand the behavior of supersymmetric noncommutative gauge theories. For these theories, the linear D-particle potential cancels, leaving a velocity dependent potential such as v^2/r^3 or v^4/r^7 for maximal supersymmetry. These potentials will still be instantaneous, so while their effects may not be as dramatic as in the nonsupersymmetric cases, they will still give rise to behavior that is unusual from the point of view of relativistic field theory.

In the maximally supersymmetric case, there is a nice interpretation of the effects of the instantaneous potential. It is well known that in the BFSS model, the leading order one-loop matrix theory potential (v^4/r^7 term and its F^4 generalization) reproduces the effects of a single supergraviton exchange in DLCQ eleven dimensional supergravity [18]. In the type IIA picture, the model describes D0-branes in a low-energy, non-relativistic limit, and the one-loop matrix model potential gives precisely the leading effects of linearized type IIA supergravity in the nonrelativistic limit. Thus, we expect that the leading nonplanar one loop effective action in maximally suspersymmetric $(2p+1)$-dimensional noncommutative gauge theory will reproduce the effects of nonrelativistic type IIA supergravity. For example, the theory will contain instantaneous 10-dimensional gravitational forces between any two sources of stress-energy.[9] We should note that the existence of gravitational forces in maximally supersymmetric noncommutative field theories has been emphasized previously in the context of the IKKT matrix model by [20]. Similar forces were discovered in [25] in the context of "nonrelativistic closed/wound string theories" and discussed in detail in [24].

A final interesting point is the claim [23] that beyond some critical temperature, the supersymmetric theory develops a perturbative instability similar to that in the bosonic theory. It would be useful to understand this directly from the matrix model perspective.

7. Comments

In this paper, we have shown that the leading infrared singular terms in the nonplanar effective action of noncommutative gauge theory correspond to instantaneous linear potentials between the D-particles making up the noncommutative space. As expected, the potential is attractive with more (adjoint) bosonic than fermionic degrees of freedom and repulsive with more fermionic degrees of freedom. In the repulsive case, the potential is positive definite, density fluctuations are confined, and the theory appears to be stable. In the attractive case, the potential is unbounded below, so the theory is unstable. In particular, pure noncommutative gauge theory is not a good quantum field theory.

We have focused on the case of maximal rank noncommutativity in $2p+1$ dimensions. However, a similar intuitive picture exists in the general case of a $d+1$ dimensional noncommutative gauge theory with θ of rank $2p$. In this case, the lower dimensional brane picture involves a system of generalized $(d-2p)$-branes with $2p$ transverse scalars taken in a background which realizes the algebra of noncommutative R^{2p}. The leading infrared singular terms will now correspond to instantaneous r^{d-2p} potentials between the $(d-2p)$-branes. Again, we expect the bosonic case to be unstable and the case with excess fermions to be stable.

[9]This has been realized independently by Shiraz Minwalla.

The description of noncommutative gauge theories in terms of D-particles also offers the following useful perspective on UV-IR mixing. The number of degrees of freedom for a system of N D-particles is proportional to N^2. For finite volume configurations with a uniform density of D-particles (for example, the fuzzy sphere) the number of degrees of freedom will therefore be finite and proportional to the volume squared. The number of degrees of freedom per unit volume is then proportional to the volume of the space, and only becomes infinite for an infinite volume distribution of D-particles. Thus, it is clear that UV divergences (associated with having an infinite number of degrees of freedom per unit volume) in maximal rank noncommutative field theory can only arise for infinite volume spaces, and that this may be attributed to the peculiar non-extensivity of the number of degrees of freedom associated with a distribution of D-particles ($d.o.f. \propto V^2$). UV-IR mixing in manifest, since the UV degrees of freedom involve the large "strings" connecting distant D-particles (this is consistent with previous identifications of the UV degrees of freedom with "bi-local fields" [12] or "stretched strings" [21]).

Finally, we note that the contribution to the leading IR singular terms in the nonplanar effective action was completely independent of the particle masses in the field theory formalism. In terms of the D-particle language, adding masses to the matter in the theory will affect the form of the potential between D-particles at short (or intermediate) distances but the long distance form of the potential will remain linear. Thus, the overall stability of the theory does not depend on the relative number of massless bosons and fermions, but rather on the relative number at all scales. Put another way, the tendency of the noncommutative space in its perturbative vacuum state to contract, expand, or remain fixed as a result of the zero point energies of fluctuating fields is determined only by the relative number of bosons and fermions at high energies and is independent of particle masses. It would be interesting if similar behavior, in particular the insensitivity to low scale supersymmetry breaking of the long-range effects of vacuum fluctuations, exists in other theories whose number of degrees of freedom is not proportional to volume.[10]

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[10]We thank Tom Banks for pointing out that this insensitivity does not seem to hold for theories with supersymmetry broken by Scherk-Schwarz boundary conditions such as those in [22].

A. Derivation of the quadratic effective action

In this section, we compute the leading IR singular term in the nonplanar quadratic effective action for noncommutative gauge theory in d dimensions. This calculation was performed in the four dimensional case in [5, 6].

To determine the leading term we are interested in, it is enough to compute the contribution from a single scalar field. As shown in [6], the gauge field and fermion contributions take precisely the same form, and have coefficients such that the leading term cancels in the supersymmetric case. For calculational purposes, we work in d-dimensional Euclidean space with the action

$$S = \int d^d x \left(\frac{1}{4} F_{ab} F_{ab} + \frac{1}{2} D_a \phi D_a \phi + \frac{1}{2} m^2 \phi^2 \right) ,$$

where

$$D_a \phi = \partial_a \phi + ig[A_a, \phi]_\star \, .$$

The effective action obtained by integrating out the scalar is given by (star products are implied)

$$e^{-\Gamma} = \langle e^{-\int d^d x (ig\partial_a \phi [A_a, \phi] - \frac{g^2}{2}[A_a,\phi][A_a,\phi])} \rangle \, ,$$

where we treat A as a background field and simply perform the gaussian integral over ϕ. At quadratic order in A, we have

$$\Gamma = -\frac{g^2}{2} \langle \int d^d x (i\partial_a \phi [A_a, \phi]) \int d^d y (i\partial_a \phi [A_a, \phi]) \rangle - \\ -\frac{g^2}{2} \langle \int d^d x [A_a, \phi][A_a, \phi] \rangle \, .$$

Using the momentum space propagator

$$\langle \phi(k) \phi(l) \rangle = (2\pi)^d \delta^d(k + l) \frac{1}{k^2 + m^2} \, ,$$

we find that the nonplanar parts of this expectation value give

$$\begin{aligned} \Gamma_{NP} = g^2 \int \frac{d^d k}{(2\pi)^d} A_a(k) A_b(-k) \int \frac{d^d l}{(2\pi)^d} l_a l_b \frac{e^{ik \times l}}{(l^2 + m^2)((l+k)^2 + m^2)} - \\ -g^2 \int \frac{d^d k}{(2\pi)^d} A_a(k) A_b(-k) \int \frac{d^d l}{(2\pi)^d} \frac{e^{ik\times l}}{(l^2 + m^2)} \\ = g^2 \int \frac{d^d k}{(2\pi)^d} A_a(k) A_b(-k) \{ -2\partial_{\tilde{k}_a} \partial_{\tilde{k}_b} + (\partial^2_{\tilde{k}} - m^2) \delta_{ab} \} \int \frac{d^d l}{(2\pi)^d} \frac{e^{ik\times l}}{(l^2 + m^2)^2} + \cdots \, , \end{aligned}$$

where $\tilde{k}^a = \theta^{ab} k_b$, and in the last line the dots indicate higher order terms arising from expanding the propagators about $k = 0$. Using

$$\frac{1}{(l^2 + m^2)^2} = \int_0^\infty d\alpha \alpha e^{-\alpha(l^2 + m^2)} \, ,$$

we may evaluate the l integral, yielding

$$\begin{aligned}
\Gamma &= \frac{g^2}{(4\pi)^{\frac{d}{2}}}\int\frac{d^dk}{(2\pi)^d}A_a(k)A_b(-k)\{-2\partial_{\tilde{k}_a}\partial_{\tilde{k}_b}+(\partial_{\tilde{k}}^2-m^2)\delta_{ab}\}\int_0^\infty\frac{d\alpha}{\alpha^{\frac{d}{2}-1}}e^{-\alpha m^2-\frac{\tilde{k}^2}{4\alpha}}\\
&= \frac{g^2}{(4\pi)^{\frac{d}{2}}}\int\frac{d^dk}{(2\pi)^d}A_a(k)A_b(-k)\int_0^\infty\frac{d\alpha}{\alpha^{\frac{d}{2}-1}}e^{-\alpha m^2-\frac{\tilde{k}^2}{4\alpha}}\times\\
&\quad\times\left(-\frac{1}{2\alpha^2}\tilde{k}_a\tilde{k}_b+\delta_{ab}\left(\frac{1}{\alpha}\left(1-\frac{d}{2}\right)+\frac{\tilde{k}^2}{4\alpha^2}-m^2\right)\right)\\
&= -\frac{g^2}{2\cdot(4\pi)^{\frac{d}{2}}}\int\frac{d^dk}{(2\pi)^d}A_a(k)A_b(-k)\tilde{k}_a\tilde{k}_b\int_0^\infty\frac{d\alpha}{\alpha^{\frac{d}{2}+1}}e^{-\alpha m^2-\frac{\tilde{k}^2}{4\alpha}}+\\
&\quad+\frac{g^2}{(4\pi)^{\frac{d}{2}}}\int\frac{d^dk}{(2\pi)^d}A_a(k)A_a(-k)\int_0^\infty d\alpha\partial_\alpha\left(\frac{1}{\alpha^{\frac{d}{2}-1}}e^{-\alpha m^2-\frac{\tilde{k}^2}{4\alpha}}\right).
\end{aligned}$$

The vanishing of the last line provides a check of the calculation so far since the A_aA_a structure is not consistent with gauge invariance. For small values of momenta (or setting m to zero, the remaining term gives our final result,

$$\Gamma_{NP}=\int\frac{d^dk}{(2\pi)^d}A_a(k)A_b(-k)\left(-\frac{g^2}{2\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)\frac{\tilde{k}^a\tilde{k}^b}{|\tilde{k}|^d}\right).$$

Including general adjoint matter content and translating to Minkowski space (by switching the sign), we recover the expression (2.2) stated in section 2.

B. Open wilson lines in the matrix theory language

For completeness, we recall here the derivation that the simplest open Wilson line operator in the field theory formalism corresponds to the zero-brane density operator in the matrix theory formalism [13]. It will be useful to recall that

$$e^{ik\cdot x}\star f(x)\star e^{-ik\cdot x}=f(x+\tilde{k}),$$

where $\tilde{k}^a=\theta^{ab}k_b$ as usual. Then starting in the field theory formalism, we have

$$\begin{aligned}
\omega(k) &= \int d^dxe^{ik\cdot x}\mathrm{tr}\left(P_\star e^{ig\int_0^1A_a(x+\sigma\tilde{k})\tilde{k}^a d\sigma}\right)\\
&= \lim_{N\to\infty}\int d^dx\mathrm{tr}\left(P_\star e^{\sum_{n=1}^\infty\frac{ig}{N}\int_0^1A_a(x+\frac{n}{N}\tilde{k})\tilde{k}^a}\right)\star e^{ik\cdot x}\\
&= \lim_{N\to\infty}\int d^dx\mathrm{tr}\left(\prod_\star e^{\frac{ig}{N}A_a(x+\frac{n}{N}\tilde{k})\tilde{k}^a}\right)\star e^{ik\cdot x}\\
&= \lim_{N\to\infty}\int d^dx\mathrm{tr}\left(\prod_\star e^{ik\cdot x\frac{n}{N}}\star e^{\frac{ig}{N}A_a(x)\tilde{k}^a}\star e^{-ik\cdot x\frac{n}{N}}\right)\star e^{ik\cdot x}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \int d^d x \mathrm{tr}\, \left(e^{\frac{ik\cdot x}{N}} \star e^{\frac{ig}{N}A_a(x)\tilde{k}^a} \right)^N \\
&= \lim_{N\to\infty} \int d^d x \mathrm{tr}\, \left(e^{\frac{1}{N}ik_a(x^a+g\theta^{ab}A_b)} \right)^N \\
&\to (2\pi\theta)^p \mathrm{Tr}\, \left(e^{ik\cdot X} \right),
\end{aligned}
$$

where the trace in the last line denotes the full operator trace. Gauge invariance is manifest in the matrix model formalism, where the noncommutative gauge transformations are simply unitary transformations $X^i \to UX^iU^{-1}$.

C. Direct calculation of the effective action from matrix theory

In this section, we verify the suggestion (3.4) for the dominant gauge invariant structure in the nonplanar effective action by a direct calculation in Matrix Theory. We have argued that (5.1) is consistent with the expectation that there should be a linear potential between D0-branes in nonsupersymmetric matrix theory. This expectation arises from calculations performed for widely separated D-particles in a diagonal background, whereas we are presently considering D-particles in a noncommuting configuration with uniform D-particle charge spread over $2p$ spatial dimensions. It is therefore important to check directly that the linear potential (5.1) does arise in this background.

To determine the one loop effective action in the matrix quantum mechanics, we simply need to sum the zero point energies of all fluctuating modes expanded about the background we are interested in. In practice, it is again simplest to consider the theory with an additional scalar field and then integrate out the scalar. Thus, we start with the lagrangian

$$
\mathcal{L} = \mathrm{Tr}\, \left(\frac{1}{2}\dot{X}^i\dot{X}^i + \frac{1}{4}[X^i, X^j]^2 + \frac{1}{2}\dot{\Phi}\dot{\Phi} + \frac{1}{2}[X^i, \Phi]^2 \right).
$$

The mass squared matrix for Φ may be written

$$
M^2 = (X^i \otimes \mathbb{1} - \mathbb{1} \otimes X^i)^2 .
$$

The effective potential is then given by the sum of zero point energies,

$$
\begin{aligned}
V &= \frac{1}{2}\mathrm{Tr}\, \left(\sqrt{M^2} \right) \\
&= \frac{1}{2}\mathrm{Tr}\, \left(\sqrt{(X^i \otimes \mathbb{1} - \mathbb{1} \otimes X^i)^2} \right) \\
&= -\frac{1}{2\pi^{\frac{2p+1}{2}}}\Gamma\left(\frac{2p+1}{2}\right) \int d^{2p}k\ \mathrm{Tr}\, \left(e^{ik_i(X^i\otimes 1 - 1\otimes X^i)} \right) \frac{1}{|k|^{2p+1}} + \text{commutators} \\
&= -\frac{1}{2\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right) \int d^{d-1}k\ \mathrm{Tr}\, (e^{ik\cdot X})\mathrm{Tr}\, (e^{-ik\cdot X}) \frac{1}{|k|^d} + \text{commutators}.
\end{aligned}
\tag{C.1}
$$

To see that these manipulations are sensible, note that for the background $X^i = x^i$, the matrix M^2 is diagonal and all of the eigenvalues are positive. Thus, for $X^i = x^i + \theta^{ij}A_j$, there is a well defined expansion of the square root in the second line powers of A. The Fourier transformed expressions in the third and fourth line also admit well defined expansions in powers of A (see appendix B), however these imply a fully symmetrized ordering prescription (Weyl ordering) while the expression in the second line is not fully symmetrized. As we have indicated, the difference will be terms involving commutators of X^i's and these will correspond in the field theory formalism to higher order terms of the form (3.1) in the effective action involving field strengths and covariant derivatives of field strengths inserted in the Wilson line.

It is straightforward to repeat this calculation for gauge field fluctuations and fermions (it is essentially identical to the Matrix theory effective action calculations performed in [17]). In each case, the leading order potential is identical to (C.1) with additional factors of $(d-2)$ and $-\frac{1}{2}\mathcal{N}_f$ respectively. Thus, for the general case, we may write the leading order effective potential in the matrix theory formalism as

$$V = -\frac{1}{2\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)(d-2+N_s-\mathcal{N}_f)\int d^{d-1}k\rho(k)\rho(-k)\frac{1}{|k|^d}\,.$$

After a Fourier transform to position space, this gives precisely the expression (5.1) that we deduced from the field theory result.

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